Optimal Cost is Monotonic over Preconditions

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1 Introduction

Often, we can leverage preconditions in program optimization. After all, if a program need only be correct for a smaller set of inputs, then a larger set of programs meets that requirement. Given such a set of programs, we are interested in the *optimal* program; For instance, the program with minimal execution time.

I realized that the *cost of an optimal program* for some specification is monotonic over the considered precondition (i.e., set of inputs). This has some interesting implications for *superoptimizers*, as weakening the precondition does *not* forfeit optimality; Though, it *may* violate correctness, but that is often easier to determine.

For simplicity, we consider the case where execution cost is *independent* from program input. This independence holds for linear instruction sequences. (The theorem should also hold when cost is dependent on the input – under some additional assumptions – which increases proof complexity)

2 Definitions

- $\mathbb{S}$ – The program *specification*.
- $\mathsf{exec}(p, x)$ – exec executes program (or specification) p on input x, producing an output.
- $\mathsf{cost}(p)$ – The cost of executing program p on any input.
- $\equiv_X$ – Extensional equivalence over domain X defined as (forall p and q):

$$p \ \equiv_X \ q \quad \triangleq \quad \forall x \in X.[\mathsf{exec}(p, x) \equiv \mathsf{exec}(q, x)]$$

- $\mathsf{optcost}(\mathbb{S}, X)$ – The $\mathsf{cost}(p)$ of some program p, provided that both:
  - p satisfies $\mathbb{S}$ (over X): $p \equiv_X \mathbb{S}$
  - p has minimal cost (over X): $\neg\exists q.[\mathsf{cost}(q) < \mathsf{cost}(p) \ \wedge \ q \ \equiv_X \ \mathbb{S}]$

The domain X over which p is correct (w.r.t. $\mathbb{S}$) represents the *precondition*. We demonstrate:

Theorem 1 (Optimal Cost is Monotonic over Preconditions)**.**

$$X \subseteq Y \implies \mathsf{optcost}(\mathbb{S}, X) \leq \mathsf{optcost}(\mathbb{S}, Y) \qquad \text{(for any } X, Y, \mathbb{S})$$

If we *strengthen* the precondition, the optimal cost *may only decrease*. Conversely, if we *weaken* the precondition, the optimal cost *may only increase*.

3 Proof

We prove Theorem 1. For some specification $\mathbb{S}$ and preconditions X and Y,

(A) given that $X \subseteq Y$,

we show $\mathsf{optcost}(\mathbb{S}, X) \leq \mathsf{optcost}(\mathbb{S}, Y)$.

First, we define a lemma.

Lemma 1. If $p \equiv_Y q$ and $X \subseteq Y$ then $p \equiv_X q$. (for any X, Y, p, q)

This follows trivially from the definition of $\equiv_X$ (and $\subseteq$).

Following the definition of $\mathsf{optcost}$, we know:

- There exists a program p where $\mathsf{cost}(p) = \mathsf{optcost}(\mathbb{S}, X)$, which satisfies:
  - (B) p satisfies $\mathbb{S}$ (over X): $p \equiv_X \mathbb{S}$
  - (C) p has minimal cost (over X): $\neg\exists z.[\mathsf{cost}(z) < \mathsf{cost}(p) \ \wedge \ z \ \equiv_X \ \mathbb{S}]$
- There exists a program q where $\mathsf{cost}(q) = \mathsf{optcost}(\mathbb{S}, Y)$, which satisfies:
  - (D) q satisfies $\mathbb{S}$ (over Y): $q \equiv_Y \mathbb{S}$
  - (E) q has minimal cost (over Y): $\neg\exists z.[\mathsf{cost}(z) < \mathsf{cost}(q) \ \wedge \ z \ \equiv_Y \ \mathbb{S}]$

We proceed with a proof by contradiction.

> Assume $\mathsf{cost}(p) > \mathsf{cost}(q)$.
> $q \equiv_Y \mathbb{S}$ by (D), then by Lemma 1 with (A), we know $q \equiv_X \mathbb{S}$. Thus we have:
>
> $$\mathsf{cost}(q) < \mathsf{cost}(p) \ \wedge \ q \equiv_X \mathbb{S}$$
>
> However, by (C), q cannot exist – as p has minimal cost (over X). Hence, we reached a contradiction.

Thus $\mathsf{cost}(p) \leq \mathsf{cost}(q)$. Following their definitions, $\mathsf{optcost}(\mathbb{S}, X) \leq \mathsf{optcost}(\mathbb{S}, Y)$.

$\square$